Available at
WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT
J. Math. Anal. Appl. 279 (2003) 97–110
Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS
www.elsevier.com/locate/jmaa

Global existence and asymptotic behaviour for a nonlocal phase-field model for non-isothermal phase transitions

Jürgen Sprekels [a,*] and Songmu Zheng [b,1]

[a] *Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D-10117 Berlin, Germany*
[b] *Institute of Mathematics, Fudan University, Shanghai 200437, China*

Received 3 December 2001

Submitted by A. Friedman

Abstract

In this paper a nonlocal phase-field model for non-isothermal phase transitions with a non-conserved order parameter is studied. The paper extends recent investigations to the non-isothermal situation, complementing results obtained by H. Gajewski for the non-isothermal case for conserved order parameters in phase separation phenomena. The resulting field equations studied in this paper form a system of integro-partial differential equations which are highly nonlinearly coupled. For this system, results concerning global existence, uniqueness and large-time asymptotic behaviour are derived. The main results are proved using techniques that have been recently developed by P. Krejčí and the authors for phase-field systems involving hysteresis operators.

Keywords: Phase transitions; Nonlocal models; Initial-boundary value problems; A priori estimates; Asymptotic behaviour; Well-posedness; Integrodifferential equations

1. Introduction

In a number of recent papers (see, for instance, [1,4] and the references given therein), integrodifferential (nonlocal) models for isothermal phase transitions with either conserved or non-conserved order parameters have been studied, leading to a number of results

* Corresponding author.
E-mail addresses: sprekels@wias-berlin.de (J. Sprekels), szheng@fudan.ac.cn (S. Zheng).
1 S. Zheng was partially supported by grant No. 19831060 from NSF of China.

0022-247X/03/$ – see front matter
doi:10.1016/S0022-247X(02)00559-0

concerning existence, uniqueness, and asymptotic behaviour of solutions. In the recent paper [3], the more difficult *non-isothermal* case for a conserved order parameter in phase separation has been treated. In this paper, we aim to complement the results of [3] by investigating the non-isothermal case with a non-conserved order parameter. To give a complete description of the corresponding mathematical problem, consider non-isothermal phase transitions with a non-conserved order parameter $\chi \in [0,1]$ occurring in a thermally insulated container $\Omega \subset \mathbb{R}^3$ that forms an open and bounded domain with smooth boundary $\partial\Omega$. If we denote $\Omega_T := \Omega \times (0,T)$, where $T > 0$ is some final time, and if $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, then the resulting model equations have the form

$$\mu(\theta)\chi_t = -F_1'(\chi) - \left(\frac{\beta_1}{\theta} + \beta_2\right)F_2'(\chi) - \frac{F_3'(\chi)}{\theta} - \frac{w}{\theta}, \quad \text{in } \Omega_T, \tag{1.1}$$

$$w(x,t) = \int_\Omega K\big(|x-y|\big)\big(1 - 2\chi(y,t)\big)\,dy, \quad \text{in } \Omega_T, \tag{1.2}$$

$$C_V\theta_t + \big(\beta_1 F_2'(\chi) + F_3'(\chi) + w\big)\chi_t - \kappa\Delta\theta = 0, \quad \text{in } \Omega_T, \tag{1.3}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.4}$$

$$\chi(\cdot,0) = \chi_0, \quad \theta(\cdot,0) = \theta_0, \quad \text{in } \Omega. \tag{1.5}$$

System (1.1)–(1.5) forms an initial-boundary value problem for a system in which an ordinary integrodifferential equation is coupled to a parabolic differential equation. It is the aim of this work to prove results concerning its well-posedness and large-time asymptotic behaviour (see Theorems 2.1 and 3.1 below).

Before going into mathematical details, we give a brief derivation of system (1.1)–(1.5). To this end, let θ denote the (positive) absolute temperature, and suppose that the order parameter χ represents the local volume fraction (concentration) of one of the phases, say, of the high temperature phase. For instance, if a solid-liquid transition is considered, the sets $\{\chi = 0\}$, $\{\chi = 1\}$, and $\{0 < \chi < 1\}$, correspond to solid, liquid, and mushy region, in that order. We start from the nonlocal free energy density

$$\begin{aligned} F(\chi,\theta) = {} & C_V\theta\big(1 - \ln(\theta)\big) + \theta F_1(\chi) + (\beta_1 + \beta_2\theta)F_2(\chi) + F_3(\chi) \\ & + \chi \int_\Omega K\big(|y-x|\big)\big(1 - \chi(y)\big)\,dy. \end{aligned} \tag{1.6}$$

Here, C_V (specific heat) and β_1, β_2 are positive constants, and $K : (0,\infty) \to (0,\infty)$ is a nonnegative kernel function. The functions F_1, F_3 are smooth where F_3 is usually concave (often a linear function or a quadratic function having a negative leading term); moreover, F_2 is a convex function that acts as a barrier, i.e., forces the concentration χ to attain values in the physically meaningful range $[0,1]$. Typical choices are $F_1(\chi) = -L\chi/\theta_c$, $F_3(\chi) = L\chi$, where $L > 0$ and $\theta_c > 0$ represent latent heat of phase transition and phase transition temperature, respectively, while F_2 is given by either $F_2(\chi) = \chi\ln(\chi) + (1-\chi)\ln(1-\chi)$ or

$$F_2(\chi) = I_{[0,1]}(\chi) = \begin{cases} 0, & \text{if } \chi \in [0,1], \\ +\infty, & \text{otherwise,} \end{cases}$$

In this paper, we study the differentiable logarithmic case; the case of the merely subdifferentiable indicator function, in which the system corresponding to (1.1)–(1.5) can be viewed as a nonlocal version of a relaxed Stefan problem of Penrose–Fife type (cf. [2,9]), will be the subject of the forthcoming paper [7].

Following the rules of thermodynamics, we introduce the densities of entropy S and internal energy E by

$$\begin{aligned} S(\chi,\theta) &= -\partial_\theta F(\chi,\theta) = C_V \ln(\theta) - F_1(\chi) - \beta_2 F_2(\chi), \\ E(\chi,\theta) &= F(\chi,\theta) + \theta S(\chi,\theta) = C_V\theta + \beta_1 F_2(\chi) + F_3(\chi) \\ &\qquad + \chi \int_\Omega K\big(|x-y|\big)\big(1-\chi(y)\big)\,dy. \end{aligned} \tag{1.7}$$

To find equilibrium values for χ and θ, we maximize the total entropy functional

$$\mathcal{S}[\chi,\theta] := \int_\Omega S(\chi,\theta)\,dx = \int_\Omega \big(C_V \ln(\theta) - F_1(\chi) - \beta_2 F_2(\chi)\big)\,dx \tag{1.8}$$

under the constraint that total internal energy be conserved, i.e., that

$$\begin{aligned} \mathcal{E}[\chi,\theta] := \int_\Omega E(\chi,\theta)\,dx = \int_\Omega \bigg(& C_V\theta + \beta_1 F_2(\chi) + F_3(\chi) \\ & + \chi \int_\Omega K\big(|x-y|\big)\big(1-\chi(y)\big)\,dy \bigg)\,dx = \text{const.} \end{aligned} \tag{1.9}$$

Applying Lagrange's method, we maximize the augmented entropy

$$\mathcal{S}_\lambda[\chi,\theta] := \mathcal{S}[\chi,\theta] + \lambda\mathcal{E}[\chi,\theta], \tag{1.10}$$

which leads to the Euler–Lagrange equations

$$\begin{aligned} \partial_\chi \mathcal{S}_\lambda &= -F_1'(\chi) + (\lambda\beta_1 - \beta_2)F_2'(\chi) + \lambda F_3'(\chi) + \lambda w = 0, \\ \partial_\theta \mathcal{S}_\lambda &= \frac{C_V}{\theta} + \lambda C_V = 0, \end{aligned} \tag{1.11}$$

with w given by (1.2). From the second identity in (1.11) the Lagrange multiplier is easily identified as $\lambda = -1/\theta$.

We now postulate that the evolution of χ runs in the direction of $\partial_\chi \mathcal{S}_\lambda$ at a rate which is proportional to it. That is, the evolution of χ is governed by the evolution equation $\mu(\theta)\chi_t = \partial_\chi \mathcal{S}_\lambda[\chi,\theta]$ which is identical to (1.1).

The evolution of θ is described by the balance of internal energy which in the absence of distributed sources becomes

$$E_t + \nabla\cdot\mathbf{q} = 0. \tag{1.12}$$

Under the assumption $\mathbf{q} = -\kappa\nabla\theta$, where $\kappa > 0$ denotes the constant heat conductivity, we obtain (1.3) as energy balance.

Next, we study the thermodynamic consistency of the model. Assuming that $\theta > 0$ (which will have to be verified below), we obtain from a straightforward calculation, using (1.1), (1.12), and the boundary condition (1.4), that

$$\int_\Omega \left[\frac{dS}{dt}(\chi,\theta) + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right)\right]dx = \int_\Omega \left[\frac{dS}{dt}(\chi,\theta) - \frac{1}{\theta}\frac{dE}{dt}(\chi,\theta) + \frac{\kappa}{\theta^2}|\nabla\theta|^2\right]dx$$
$$= \int_\Omega \left[\frac{\kappa}{\theta^2}|\nabla\theta|^2 + \mu(\theta)\chi_t^2\right]dx \geqslant 0. \tag{1.13}$$

Therefore, the Clausius–Duhem inequality is satisfied in integrated form which means that our model complies with the Second Principle of Thermodynamics.

The main mathematical novelties of the results stated below in comparison to other non-isothermal phase-field models for non-conserved order parameters lie in the occurrence of the integral expression w in the equations and in the fact that in (1.1) the singular term $F_2'(\chi)$ occurs while no diffusive term is present. This entails a loss of spatial smoothness of the unknown χ so that the line of argumentation based on Moser-type iteration techniques which has been developed in [5] for the local case in a similar context cannot be employed. On the other hand, (1.1) is an ordinary integrodifferential equation, so that ODE-techniques can be used, and the integral expression (1.2) has a smoothing effect. It will turn out that these two advantages counterbalance the loss in spatial smoothness of χ.

The remainder of the paper is organized as follows: In Section 2, we give a detailed statement of the mathematical problem, and we prove global existence and uniqueness. In the final Section 3, the asymptotic behaviour as $t \to +\infty$ is studied.

In what follows, the norms of the standard Lebesgue spaces $L^p(\Omega)$, for $1 \leqslant p \leqslant \infty$, will be denoted by $\|\cdot\|_p$. Finally, we shall use the usual denotations $W^{m,p}(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, for the standard Sobolev spaces.

2. Global existence and uniqueness

Consider the problem (1.1)–(1.5). For the sake of a simpler notation, we normalize $C_V = \kappa = 1$ which has no bearing on the mathematical analysis. We make the following general assumptions on the data of our system.

(H1) $\chi_0 \in L^\infty(\Omega)$, $\theta_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, and there are positive constants a_0, b_0, δ such that $0 < a_0 \leqslant \chi_0(x) \leqslant b_0 < 1$ and $\theta_0(x) \geqslant \delta > 0$ for a.e. $x \in \Omega$.

(H2) $F_i \in C^2[0,1]$, $i = 1, 3$, and $F_2 \in C^2(0,1)$ is such that F_2' is strictly increasing on (0,1) and that

$$\lim_{\chi\searrow 0} F_2'(\chi) = -\infty, \qquad \lim_{\chi\nearrow 1} F_2'(\chi) = +\infty. \tag{2.1}$$

We denote by $G : \mathbb{R} \to (0,1)$, $G \in C^1(\mathbb{R})$, the inverse of F_2'.

(H3) $\mu \in C^1(0,+\infty)$, and there is some $\hat{\mu} > 0$ such that

$$\mu(\theta) \geqslant \hat{\mu}\min\{\theta^{-1}, 1\} \quad \forall\theta > 0. \tag{2.2}$$

(H4) The kernel function K is nonnegative on its domain of definition and so smooth that the linear integral operator $\chi \mapsto \mathcal{P}[\chi]$,

$$\mathcal{P}[\chi](x) := \int_{\Omega} K\big(|x-y|\big)\chi(y)\,dy, \quad x \in \Omega, \tag{2.3}$$

is defined on $L^2(\Omega)$, maps bounded subsets of $L^\infty(\Omega)$ into bounded subsets of $L^\infty(\Omega)$, and has the following continuity property:

$$\begin{aligned} &\text{If } \{\chi_k\}_{k\in\mathbb{N}} \subset H^1\big(0,T;L^2(\Omega)\big) \quad \text{is such that } \chi_{k,t} \to \chi_t \\ &\text{strongly in } L^2(\Omega_T) \quad \text{and } \chi_k \to \chi \quad \text{weakly-* in } L^\infty(\Omega_T), \\ &\text{then } \mathcal{P}[\chi_k] \to \mathcal{P}[\chi] \quad \text{weakly in } L^2(\Omega_T). \end{aligned} \tag{2.4}$$

(H5) $\beta_1 > 0$, $\beta_2 > 0$.

Remark 1. The assumptions on F_2 are obviously satisfied for the case that $F_2(\chi) = \chi\ln(\chi) + (1-\chi)\ln(1-\chi)$. Under the assumption (H5), and for suitable choices of F_1, F_3, the free energy then becomes the Flory–Huggins free energy arising in the theory of polymers.

Remark 2. We stress the fact that for our analysis below to work it is crucial that β_1 and β_2 are positive. However, this assumption seems to be natural from physical reasons.

Remark 3. Hypothesis (H3) is satisfied if $\mu(\theta) = \hat{\mu}\theta^{\alpha-1}$ for some $\alpha \in [0,1]$. Note that for $\alpha = 1$ a nonlocal analogue to a phase-field system of Penrose–Fife type with zero interfacial energy results, while for $\alpha = 0$ we obtain a nonlocal analogue of the Caginalp model with zero interfacial energy.

We aim to prove the following general existence result:

Theorem 2.1. *Suppose that the general hypotheses* (H1)–(H5) *hold. Then system* (1.1)–(1.5) *admits a unique solution* $(\chi,\theta) \in (L^\infty(\Omega_T))^2$ *such that*

(i) $\chi \in H^2(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$, $\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$.
(ii) $0 < \chi < 1$ *and* $\theta > 0$ *a.e. in* Ω_T.

Moreover, (χ,θ) *has the following additional properties*:

(iii) *There are constants* $0 < a_1 < b_1 < 1$, *independent of* T, *such that* $a_1 \leqslant \chi(x,t) \leqslant b_1$ *a.e. in* Ω_T.
(iv) *There is a constant* $\hat{c} > 0$, *independent of* T, *such that* $\theta(x,t) \geqslant \delta e^{-\hat{c}t}$ *a.e. in* Ω_T.

Proof. The idea of the proof is as follows: We construct a suitable "cut-off" version of the system (1.1)–(1.5) which can be shown to have a unique solution having the required smoothness properties by using the same technique as in the proof of Theorem 3.1 in [6]; after that, we apply ODE barrier techniques and parabolic estimates to show that the solution to the cut-off system is in fact the unique solution to the original system (1.1)–(1.5). We divide our proof into a sequence of steps.

Step 1. Construction of a "cut-off" system.

Let $0 < \varepsilon < 1$ and $0 < \alpha < 1/2$ be constants which will be specified later. We put $p(\alpha) := \min\{\alpha, a_0\}$, $q(\alpha) := \max\{1-\alpha, b_0\}$, and define the auxiliary functions $T_\varepsilon, \mu_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ and $Z_\alpha, F_{i,\alpha} : \mathbb{R} \to \mathbb{R}$ by putting

$$\begin{aligned}
&T_\varepsilon(s) := \max\bigl\{\varepsilon, |s|\bigr\}, \quad \mu_\varepsilon(s) := \mu\bigl(T_\varepsilon(s)\bigr), \quad \text{for } s \in \mathbb{R},\\
&F_{i,\alpha}(s) := \begin{cases} F_i(p(\alpha)) + F_i'(p(\alpha))(s - p(\alpha)), & s \leqslant p(\alpha),\\ F_i(s), \quad p(\alpha) \leqslant s \leqslant q(\alpha), & i = 1,2,3,\\ F_i(q(\alpha)) + F_i'(q(\alpha))(s - q(\alpha)), & s \geqslant q(\alpha),\end{cases}\\
&Z_\alpha(s) := \begin{cases} p(\alpha), & s \leqslant p(\alpha),\\ s, & p(\alpha) \leqslant s \leqslant q(\alpha),\\ q(\alpha), & s \geqslant q(\alpha).\end{cases}
\end{aligned} \tag{2.5}$$

We note the following facts:

(i) $T_\varepsilon' \in L^\infty(\mathbb{R})$; $Z_\alpha \in W^{1,\infty}(\mathbb{R})$.
(ii) $F_{i,\alpha}' \in W^{1,\infty}(\mathbb{R})$, $i = 1,2,3$.
(iii) μ_ε is Lipschitz continuous on compact subsets of $\mathbb{R}$, and from (2.2) we have the estimates

$$\begin{aligned}
&\mu_\varepsilon(\theta) \geqslant \hat{\mu}\min\bigl\{1, 1/T_\varepsilon(\theta)\bigr\} \quad \forall\theta \in \mathbb{R},\\
&\frac{1}{\mu_\varepsilon(\theta)} \leqslant \frac{1}{\hat{\mu}}\max\bigl\{1, T_\varepsilon(\theta)\bigr\} \leqslant \frac{1}{\hat{\mu}}\bigl(1 + |\theta|\bigr) \quad \forall\theta \in \mathbb{R},\\
&\frac{1}{T_\varepsilon(\theta)\mu_\varepsilon(\theta)} \leqslant \frac{1}{\hat{\mu}}\max\bigl\{1, 1/T_\varepsilon(\theta)\bigr\} \leqslant \frac{1}{\varepsilon\hat{\mu}} \quad \forall\theta \in \mathbb{R}.
\end{aligned} \tag{2.6}$$

With the above functions, we consider the following "cut-off" version of system (1.1)–(1.5):

$$\mu_\varepsilon(\theta)\chi_t = -F_{1,\alpha}'(\chi) - \left(\frac{\beta_1}{T_\varepsilon(\theta)} + \beta_2\right)F_{2,\alpha}'(\chi) - \frac{F_{3,\alpha}'(\chi)}{T_\varepsilon(\theta)} - \frac{w_\alpha}{T_\varepsilon(\theta)}, \quad \text{in } \Omega_T, \tag{2.7}$$

$$w_\alpha(x,t) = \int_\Omega K\bigl(|x-y|\bigr)\bigl(1 - 2Z_\alpha\bigl(\chi(y,t)\bigr)\bigr)\,dy, \quad \text{in } \Omega_T, \tag{2.8}$$

$$\theta_t + \bigl(\beta_1 F_{2,\alpha}'(\chi) + F_{3,\alpha}'(\chi) + w_\alpha\bigr)\chi_t - \Delta\theta = 0, \quad \text{in } \Omega_T, \tag{2.9}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{2.10}$$

$$\chi(\cdot,0) = \chi_0, \quad \theta(\cdot,0) = \theta_0, \quad \text{in } \Omega. \tag{2.11}$$

We claim that the system (2.7)–(2.11) admits a unique solution $(\chi^{\varepsilon,\alpha}, \theta^{\varepsilon,\alpha}) \in (L^\infty(\Omega_T))^2$, with $\chi_t^{\varepsilon,\alpha} \in L^\infty(\Omega_T)$ and $\theta^{\varepsilon,\alpha} \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, such that (2.7)–(2.9) hold a.e. in Ω_T, (2.10) a.e. on $\partial\Omega \times (0,T)$, and (2.11) a.e. in Ω. To prove this assertion, we employ the same technique as in the proof of Theorem 3.1 in [6]. In fact, we have tailored the cut-off system (2.7)–(2.11) in such a way that this technique works. Since the line of argumentation is very similar and can be carried over in a straightforward manner with only minor and quite obvious modifications which are caused by the nonlocal term w_α, we can afford to only sketch the details, here.

The idea is to use successive approximation. To this end, put $\theta^0(x,t) := \theta^0(x)$ for $(x,t) \in \Omega_T$, and define for $k \in \mathbb{N}$ the iterate (χ^k, θ^k) as the unique solution to the initial boundary problem

$$\chi_t^k = -\frac{1}{\mu_\varepsilon(\theta^{k-1})}\left[F'_{1,\alpha}(\chi^k) + \left(\frac{\beta_1}{T_\varepsilon(\theta^{k-1})} + \beta_2\right)F'_{2,\alpha}(\chi^k) + \frac{F'_{3,\alpha}(\chi^k)}{T_\varepsilon(\theta^{k-1})} + \frac{w_\alpha^k}{T_\varepsilon(\theta^{k-1})}\right], \quad \text{in } \Omega_T, \tag{2.12}$$

$$w_\alpha^k(x,t) = \int_\Omega K\big(|x-y|\big)\big(1 - 2Z_\alpha\big(\chi^k(y,t)\big)\big)\,dy, \text{ in } \Omega_T, \tag{2.13}$$

$$\chi^k(x,0) = \chi^0(x), \quad x \in \Omega, \tag{2.14}$$

$$\theta_t^k - \Delta\theta^k + \theta^k = \theta^{k-1} - \big[\beta_1 F'_{2,\alpha}(\chi^k) + F'_{3,\alpha}(\chi^k) + w_\alpha^k\big]\chi_t^k, \quad \text{in } \Omega_T, \tag{2.15}$$

$$\frac{\partial\theta^k}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.16}$$

$$\theta^k(x,0) = \theta^0(x), \quad x \in \Omega. \tag{2.17}$$

Note that if $\theta^{k-1} \in L^\infty(\Omega_T)$ is known then (2.12)–(2.14) is an initial value problem for an ordinary integrodifferential equation containing only bounded nonlinearities in χ which are globally Lipschitz continuous (in particular, the integral operator defined in (2.13) is globally bounded and Lipschitz continuous on $L^\infty(\Omega_T)$). Hence, (2.12)–(2.14) has a unique global solution $\chi^k \in W^{1,\infty}(0,T;L^\infty(\Omega))$. But then (2.15)–(2.17) constitutes a linear heat conduction problem, where the right-hand side of (2.15) belongs to $L^\infty(\Omega_T)$. Using standard parabolic theory (cf., for instance, Lemma 3.3 in [6]), we can infer that (2.15)–(2.17) admits a unique solution $\theta^k \in L^\infty(\Omega_T) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, so that the iterative procedure (2.12)–(2.17) is well-defined and produces a sequence $(\chi^k,\theta^k) \in (L^\infty(\Omega_T))^2$, where $\chi_t^k \in L^\infty(\Omega_T)$ and $\theta^k \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Next, observe that (2.6) and the boundedness of the nonlinear terms on the right-hand side of (2.12) imply the existence of some $C_1 > 0$ (which is independent of $k \in \mathbb{N}$) such that

$$\big|\chi_t^k(x,t)\big| \leqslant C_1\big(1 + |\theta^{k-1}(x,t)|\big) \quad \text{a.e. in } \Omega_T. \tag{2.18}$$

Therefore, using the global boundedness of the terms in the bracket on the right-hand side of (2.15) which multiplies χ_t^k, we obtain from standard parabolic estimates (cf. Lemma 3.3 in [6], again) that

$$\|\theta^k\|_{L^\infty(\Omega_T)} \leqslant C_2, \tag{2.19}$$

with some $C_2 \geqslant \|\theta_0\|_\infty$ which is independent of $k \in \mathbb{N}$. Taking C_2 larger, if necessary, we then conclude that also

$$\left\|\theta_t^k\right\|_{L^2(\Omega_T)} + \|\Delta\theta^k\|_{L^2(\Omega_T)} \leqslant C_2 \quad \forall k \in \mathbb{N}, \tag{2.20}$$

which means that $\{\theta^k\}$ is bounded in $L^\infty(\Omega_T) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Now that this is shown, we can employ the general properties (i)–(iii) and (2.6) of the cut-off functions $T_\varepsilon, \mu_\varepsilon, Z_\alpha, F_{i,\alpha}$ to show by an analogous argumentation as in the proof of Theorem 3.1 in [6] (for details, we refer to [7] where a more general situation is considered) that the following holds:

(iv) $\{\theta^k\}$ is a Cauchy sequence in $L^2(\Omega_T)$.
(v) $\{\chi^k\}$ and $\{\chi_t^k\}$ are Cauchy sequences in $L^2(\Omega_T)$.

Therefore we can claim that there exist functions $\chi^{\varepsilon,\alpha}, \theta^{\varepsilon,\alpha}$ such that the following convergences hold:

$$\begin{aligned}
&\chi_t^k \to \chi_t^{\varepsilon,\alpha}, \text{ strongly in } L^2(\Omega_T) \text{ and weakly-* in } L^\infty(\Omega_T),\\
&\chi^k \to \chi^{\varepsilon,\alpha}, \text{ strongly in } C\big([0,T];L^2(\Omega)\big) \text{ and weakly-* in } L^\infty(\Omega_T),\\
&\theta^k \to \theta^{\varepsilon,\alpha}, \text{ strongly in } L^2(\Omega_T), \text{ weakly-* in } L^\infty(\Omega_T),\\
&\qquad\qquad \text{and weakly in } H^1\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big).
\end{aligned} \tag{2.21}$$

By (H4), we then have $\mathcal{P}[\chi^k] \to \mathcal{P}[\chi^{\varepsilon,\alpha}]$, weakly in $L^2(\Omega_T)$, and letting $k \nearrow \infty$ in (2.12)–(2.17), we easily obtain that $(\chi^{\varepsilon,\alpha}, \theta^{\varepsilon,\alpha})$ is a solution to the cut-off system (2.7)–(2.11) having the asserted properties. Arguing as in the proof of Theorem 3.1 in [6], we can easily infer that $(\chi^{\varepsilon,\alpha}, \theta^{\varepsilon,\alpha})$ is the unique solution to (2.7)–(2.11) with these properties.

Step 2. Existence of a solution having the properties (i)–(iv) in Theorem 2.1.

We now aim to show that for sufficiently small $\alpha > 0$, $\varepsilon > 0$ the cut-off solution $(\chi^{\varepsilon,\alpha}, \theta^{\varepsilon,\alpha})$ is in fact a solution to (1.1)–(1.5) having the properties (i)–(iv) asserted in the statement of Theorem 2.1. To this end, we at first consider (2.7) which holds for all $(x,t) \in \Omega_T \setminus M$ where M has measure zero. In what follows, we only consider the set $\Omega_T \setminus M$. Since $\mu_\varepsilon(\theta^{\varepsilon,\alpha}) > 0$ on $\Omega_T \setminus M$, we conclude that for $(x,t) \in \Omega_T \setminus M$ we have $\chi_t^{\varepsilon,\alpha}(x,t) \geqslant 0$ if and only if

$$\begin{aligned}
F'_{2,\alpha}\big(\chi^{\varepsilon,\alpha}(x,t)\big) \leqslant{}& \frac{-T_\varepsilon(\theta^{\varepsilon,\alpha}(x,t))}{\beta_1 + \beta_2 T_\varepsilon(\theta^{\varepsilon,\alpha}(x,t))} F'_{1,\alpha}\big(\chi^{\varepsilon,\alpha}(x,t)\big)\\
&- \frac{1}{\beta_1 + \beta_2 T_\varepsilon(\theta^{\varepsilon,\alpha}(x,t))}\big(F'_{3,\alpha}\big(\chi^{\varepsilon,\alpha}(x,t)\big) + w_\alpha(x,t)\big).
\end{aligned} \tag{2.22}$$

Likewise, $\chi_t^{\varepsilon,\alpha}(x,t) \leqslant 0$ if and only if (2.22) holds with $\leqslant$ replaced by $\geqslant$. Now it holds, by construction,

$$\sup_{s\in\mathbb{R}}\big|F'_{1,\alpha}(s)\big| \leqslant \big\|F'_1\big\|_{C[0,1]}, \qquad \sup_{s\in\mathbb{R}}\big|F'_{3,\alpha}(s)\big| \leqslant \big\|F'_3\big\|_{C[0,1]}, \tag{2.23}$$

$$-\big\|\mathcal{P}[1]\big\|_\infty \leqslant w_\alpha(x,t) \leqslant \big\|\mathcal{P}[1]\big\|_\infty, \quad \text{where } \mathcal{P}[1](x) = \int_\Omega K\big(|x-y|\big)\,dy, \tag{2.24}$$

as well as

$$0 \leqslant \frac{T_\varepsilon(\theta^{\varepsilon,\alpha}(x,t))}{\beta_1 + \beta_2 T_\varepsilon(\theta^{\varepsilon,\alpha}(x,t))} \leqslant \frac{1}{\beta_2}, \qquad 0 \leqslant \frac{1}{\beta_1 + \beta_2 T_\varepsilon(\theta^{\varepsilon,\alpha}(x,t))} \leqslant \frac{1}{\beta_1}. \tag{2.25}$$

Therefore, the absolute value of the right-hand side of (2.22) is bounded from above by the finite constant

$$\hat{\gamma} := \frac{1}{\beta_2}\|F_1'\|_{C[0,1]} + \frac{1}{\beta_1}\big(\|F_3'\|_{C[0,1]} + \|\mathcal{P}[1]\|_\infty\big) \tag{2.26}$$

which is independent of α, ε and t. Consequently, we have $\chi_t^{\varepsilon,\alpha}(x,t) \geqslant 0$ if $F'_{2,\alpha}(\chi^{\varepsilon,\alpha}(x,t)) \leqslant -\hat{\gamma}$, and $\chi_t^{\varepsilon,\alpha}(x,t) \leqslant 0$ if $F'_{2,\alpha}(\chi^{\varepsilon,\alpha}(x,t)) \geqslant \hat{\gamma}$. We now fix some $\hat{\alpha} > 0$ which is so small that $p(\hat{\alpha}) \leqslant G(-\hat{\gamma})$ and $q(\hat{\alpha}) \geqslant G(\hat{\gamma})$, where G is the (strictly increasing) inverse of F_2' (recall (H2)).

If then $\chi^{\varepsilon,\hat{\alpha}}(x,t) < p(\hat{\alpha})$, it follows that $F'_{2,\hat{\alpha}}(\chi^{\varepsilon,\hat{\alpha}}(x,t)) = F_2'(p(\hat{\alpha})) \leqslant -\hat{\gamma}$ so that $\chi_t^{\varepsilon,\hat{\alpha}}(x,t) \geqslant 0$. Likewise, if $\chi^{\varepsilon,\hat{\alpha}}(x,t) > q(\hat{\alpha})$ then $\chi_t^{\varepsilon,\hat{\alpha}}(x,t) \leqslant 0$. In conclusion, we have for a.e. $(x,t) \in \Omega_T$ the inequality

$$a_1 := p(\hat{\alpha}) = \min\{\hat{\alpha}, a_0\} \leqslant \chi^{\varepsilon,\hat{\alpha}}(x,t) \leqslant b_1 := q(\hat{\alpha}) = \max\{1-\hat{\alpha}, b_0\}. \tag{2.27}$$

Note that $0 < a_1 < b_1 < 1$, and the constants a_1, b_1 are independent of ε and T. Besides, denoting $(\chi^\varepsilon, \theta^\varepsilon) := (\chi^{\varepsilon,\hat{\alpha}}, \theta^{\varepsilon,\hat{\alpha}})$, we have the identities

$$\begin{aligned}
&F_{i,\hat{\alpha}}(\chi^\varepsilon) = F_i(\chi^\varepsilon), \quad i = 1,2,3,\\
&Z_{\hat{\alpha}}(\chi^\varepsilon) = \chi^\varepsilon, \quad \text{a.e. in } \Omega_T,\\
&w_{\hat{\alpha}}(x,t) = w(x,t) = \int_\Omega K\big(|x-y|\big)\big(1 - 2\chi^\varepsilon(y,t)\big)\,dy, \quad \text{a.e. in } \Omega_T.
\end{aligned} \tag{2.28}$$

Therefore, the pair $(\chi^\varepsilon, \theta^\varepsilon)$ satisfies Eqs. (1.2)–(1.5), and we have

$$\mu_\varepsilon(\theta^\varepsilon)\chi_t^\varepsilon = -F_1'(\chi^\varepsilon) - \left(\frac{\beta_1}{T_\varepsilon(\theta^\varepsilon)} + \beta_2\right)F_2'(\chi^\varepsilon) - \frac{F_3'(\chi^\varepsilon)}{T_\varepsilon(\theta^\varepsilon)} - \frac{w}{T_\varepsilon(\theta^\varepsilon)}, \quad \text{a.e. in } \Omega_T. \tag{2.29}$$

We now aim to show that there is some $\hat{\varepsilon} > 0$ such that $\theta^{\hat{\varepsilon}}(x,t) \geqslant \hat{\varepsilon}$ a.e. in Ω_T. It then follows that $T_{\hat{\varepsilon}}(\theta^{\hat{\varepsilon}}) = \theta^{\hat{\varepsilon}}$ and thus $\mu_{\hat{\varepsilon}}(\theta^{\hat{\varepsilon}}) = \mu(\theta^{\hat{\varepsilon}})$ which then implies that $(\chi^{\hat{\varepsilon}}, \theta^{\hat{\varepsilon}})$ also satisfies (1.1), i.e., is a solution to (1.1)–(1.5).

To this end, we test Eq. (1.3) by an arbitrary function $p \in H^1(\Omega_T)$ satisfying $p \leqslant 0$ a.e. in Ω_T. Putting $z := \beta_1 F_2'(\chi) + F_3'(\chi) + w$, we obtain

$$\int_\Omega \big(p\theta_t^\varepsilon + \nabla p \cdot \nabla\theta^\varepsilon\big)(x,t)\,dx = \int_\Omega \big(|p| z \chi_t^\varepsilon\big)(x,t)\,dx. \tag{2.30}$$

We have, by (2.29),

$$z\chi_t^\varepsilon = -\frac{1}{\mu_\varepsilon(\theta^\varepsilon)} z \left(F_1'(\chi^\varepsilon) + \beta_2 F_2'(\chi^\varepsilon) + \frac{1}{T_\varepsilon(\theta^\varepsilon)} z\right). \tag{2.31}$$

We consider two cases:

Case 1. Suppose that $T_\varepsilon(\theta^\varepsilon) \leqslant 1$. Then it follows from (2.31) and (2.6), using Young's inequality, that

$$\begin{aligned} z\chi_t^\varepsilon &\leqslant \frac{T_\varepsilon(\theta^\varepsilon)}{4\mu_\varepsilon(\theta^\varepsilon)}\big(F_1'(\chi^\varepsilon) + \beta_2 F_2'(\chi^\varepsilon)\big)^2 \\ &\leqslant \frac{1}{4\hat{\mu}}\big(F_1'(\chi^\varepsilon) + \beta_2 F_2'(\chi^\varepsilon)\big)^2 T_\varepsilon(\theta^\varepsilon). \end{aligned} \tag{2.32}$$

Case 2. Let $T_\varepsilon(\theta^\varepsilon) > 1$. Then, using the second estimate in (2.6), we can infer that

$$\begin{aligned} z\chi_t^\varepsilon &\leqslant \frac{1}{\mu_\varepsilon(\theta^\varepsilon)}|z|\big|F_1'(\chi^\varepsilon) + \beta_2 F_2'(\chi^\varepsilon)\big| \\ &\leqslant \frac{1}{2\hat{\mu}}\big[\big(F_1'(\chi^\varepsilon) + \beta_2 F_2'(\chi^\varepsilon)\big)^2 + z^2\big] T_\varepsilon(\theta^\varepsilon). \end{aligned} \tag{2.33}$$

In conclusion, we always have

$$z\chi_t^\varepsilon \leqslant \frac{1}{2\hat{\mu}}\big[\big(F_1'(\chi^\varepsilon) + \beta_2 F_2'(\chi^\varepsilon)\big)^2 + z^2\big] T_\varepsilon(\theta^\varepsilon). \tag{2.34}$$

By (2.27), we therefore find the estimate $z\chi_t^\varepsilon \leqslant \hat{c} \cdot T_\varepsilon(\theta^\varepsilon)$, where the finite positive constant

$$\hat{c} := \frac{1}{2\hat{\mu}} \max_{a_1 \leqslant \chi \leqslant b_1} \big[\big(F_1'(\chi) + \beta_2 F_2'(\chi)\big)^2 + \big(\beta_1 F_2'(\chi) + F_3'(\chi) + \|\mathcal{P}[1]\|_\infty\big)^2\big] \tag{2.35}$$

is independent of ε and t. Hence, by (2.30),

$$\int_\Omega \big(p\theta_t^\varepsilon + \nabla p \cdot \nabla\theta^\varepsilon\big)(x,t)\,dx \leqslant \hat{c} \int_\Omega \big(|p| T_\varepsilon(\theta^\varepsilon)\big)(x,t)\,dx, \quad \text{a.e. in } \Omega. \tag{2.36}$$

Now put $\hat{\varepsilon} := \delta e^{-\hat{c}T}$, and

$$p(x,t) := -\big(\delta e^{-\hat{c}t} - \theta^{\hat{\varepsilon}}(x,t)\big)^+, \quad (x,t) \in \Omega_T. \tag{2.37}$$

Then we can infer from (2.36) that

$$\int_\Omega \big(p(p + \delta e^{-\hat{c}t})_t\big)(x,t)\,dx \leqslant \hat{c} \int_\Omega |p|\big(|p| + \delta e^{-\hat{c}t}\big)(x,t)\,dx, \tag{2.38}$$

whence, in particular,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega p^2(x,t)\,dx \leqslant \hat{c}\int_\Omega p^2(x,t)\,dx. \tag{2.39}$$

Therefore, by Gronwall's inequality, and since $p(x,0) = 0$, $p \equiv 0$. Thus, $\theta^{\hat{\varepsilon}}(x,t) \geqslant \delta e^{-\hat{c}t} \geqslant \hat{\varepsilon}$ a.e., which concludes the proof that $(\chi,\theta) := (\chi^{\hat{\varepsilon}}, \theta^{\hat{\varepsilon}})$ is a solution to (1.1)–(1.5) which satisfies the conditions (iii), (iv) of Theorem 2.1. By construction, we also have $\chi \in W^{1,\infty}(0,T;L^\infty(\Omega))$ and $\theta \in L^\infty(\Omega_T) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$. But then it follows that the right-hand side of (1.1) belongs to $H^1(0,T;L^2(\Omega))$ so that $\chi \in H^2(0,T;L^2(\Omega))$. In conclusion, (χ,θ) has the asserted properties (i)–(iv) of Theorem 2.1.

Step 3. Conclusion of the proof.

It remains to show that *any* solution of (1.1)–(1.5) satisfying (i), (ii) automatically satisfies (iii) and (iv), as well, and the uniqueness of the solution. To this end, suppose that an arbitrary solution (χ, θ) is given such that (i) and (ii) hold. Then we have $\theta > 0$ and thus $\mu(\theta) > 0$ a.e. in Ω_T. Moreover, $0 < \chi < 1$ a.e. in Ω_T. Therefore, we can argue similarly as in the derivation of estimate (2.27) in Step 2 above to conclude that χ satisfies $a_1 \leqslant \chi \leqslant b_1$ a.e. in Ω_T. But then the argumentation in Step 2 leading to the lower bound for the temperature may be repeated as well, showing that $\theta(x,t) \geqslant \delta e^{-\hat{c}t}$ almost everywhere. Thus, we can infer that (χ, θ) coincides in fact with the "cut-off" solution $(\chi^{\hat{\varepsilon},\hat{\alpha}}, \theta^{\hat{\varepsilon},\hat{\alpha}})$ constructed in Step 2. Since this solution is unique which can be proved by a similar proof as in [6], the uniqueness result follows. The assertion of Theorem 2.1 is thus completely proved. □

Remark 4. The result of Theorem 2.1 remains valid if (1.3) is replaced by

$$C_V\theta_t + \big(\beta_1 F_2'(\chi) + F_3'(\chi) + w\big)\chi_t - \kappa\Delta\theta = \psi(x,t,\theta), \tag{1.3$'$}$$

provided the source term ψ satisfies the following conditions: $\psi : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is measurable, and there exist some $\psi_0 \in L^\infty(\Omega_T)$ and some $\Psi > 0$ such that

(i) $\theta \leqslant 0 \Rightarrow \psi(x,t,\theta) = \psi_0(x,t)$,
(ii) $\psi_0(x,t) \geqslant 0$, a.e. in Ω_T,
(iii) $|\frac{\partial\psi}{\partial\theta}(x,t,\theta)| \leqslant \Psi$ a.e. in $\Omega \times (0,T) \times \mathbb{R}$.

Indeed, the line of argumentation used above easily generalizes to include this case; for details we refer to the proof of Theorem 3.1 in [6]. We note that then the constant $\hat{c}$ constructed above must be replaced by $\hat{c} + \Psi$.

3. Asymptotic behaviour as $t \to +\infty$

Suppose that the general hypotheses (H1)–(H5) hold. Then there is a unique pair $(\chi, \theta) \in (L^\infty_{\mathrm{loc}}(0,\infty; L^\infty(\Omega)))^2$ such that

$$\begin{aligned}
&\chi \in H^2\big(0,T; L^2(\Omega)\big) \cap W^{1,\infty}\big(0,T; L^\infty(\Omega)\big) \quad \forall T > 0,\\
&\theta \in H^1\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^2(\Omega)\big) \cap C\big([0,T]; H^1(\Omega)\big) \quad \forall T > 0,
\end{aligned} \tag{3.1}$$

$$a_1 \leqslant \chi(x,t) \leqslant b_1, \quad \theta(x,t) \geqslant \delta e^{-\hat{c}t} \quad \text{a.e. in } \Omega \times (0,\infty). \tag{3.2}$$

Besides, there is some constant $K_1 > 0$ such that

$$\sup_{(x,t)\in\Omega\times(0,\infty)} \Big[\max_{1\leqslant i\leqslant 3} \big|F_i'\big(\chi(x,t)\big)\big| + \big|w(x,t)\big|\Big] \leqslant K_1. \tag{3.3}$$

Our aim is to study the asymptotic behaviour of (χ, θ) as $t \to +\infty$. The main difficulty in doing this lies in the fact that the lower bound $\delta e^{-\hat{c}t}$ for θ tends to zero as $t \to +\infty$. We have the following result:

Theorem 3.1. *Suppose that* (H1), (H2), (H4), (H5) *hold and that*

$$\mu(\theta) \geqslant \frac{\hat{\mu}}{\theta} \quad \forall \theta > 0 \quad \textit{with some } \hat{\mu} > 0. \tag{3.4}$$

Then there exists some constant $\widehat{C}_1 > 0$ *such that the solution* (χ, θ) *to* (1.1)–(1.5) *satisfies*

$$0 < \theta(x,t) \leqslant \widehat{C}_1, \quad \left|\chi_t(x,t)\right| \leqslant \widehat{C}_1, \quad \textit{a.e. in } \Omega \times (0,\infty), \tag{3.5}$$

$$\int_0^t \int_\Omega \bigl(1+\mu(\theta)\bigr)\chi_t^2 \, dx\, d\tau + \int_0^t \int_\Omega \left(1+\frac{1}{\theta^2}\right)|\nabla\theta|^2 \, dx\, d\tau \leqslant \widehat{C}_1 \quad \forall t \geqslant 0. \tag{3.6}$$

Moreover, we have

$$\lim_{t\to\infty} \bigl\|\nabla\theta(\cdot,t)\bigr\|_2 = 0. \tag{3.7}$$

Finally, if (3.4) *holds with equality then*

$$\lim_{t\to\infty} \bigl\|\chi_t(\cdot,t)\bigr\|_2 = 0. \tag{3.8}$$

Proof. In what follows, we denote by C_k, $k \in \mathbb{N}$, positive constants that may depend on the data of the system but not on $T > 0$. We proceed in a series of steps, deriving a priori estimates for (χ, θ).

Estimate 1. Consider for $t > 0$ the energy functional

$$E(t) := \int_\Omega \Bigl[\theta(x,t) + \beta_1 F_2\bigl(\chi(x,t)\bigr) + F_3\bigl(\chi(x,t)\bigr) + \chi(x,t)\int_\Omega K\bigl(|x-y|\bigr)\bigl(1-\chi(y,t)\bigr)\,dy\Bigr]\,dx. \tag{3.9}$$

Integration of (1.3) over $\Omega \times (0,t)$, where $t > 0$, gives $E(t) \leqslant E(0)$, whence, using (3.3),

$$\sup_{t\geqslant 0} \bigl\|\theta(\cdot,t)\bigr\|_1 \leqslant C_1. \tag{3.10}$$

But then (3.4) implies, in view of (1.1) and (3.3), that

$$\bigl|\chi_t(x,t)\bigr| \leqslant C_2\bigl(1+\bigl|\theta(x,t)\bigr|\bigr) \quad \text{a.e. in } \Omega \times (0,\infty). \tag{3.11}$$

Applying Theorem 3.1 in [8] yields

$$\theta(x,t) \leqslant C_3 \quad \text{a.e. in } \Omega \times (0,\infty), \tag{3.12}$$

and (3.5) is proved.

Estimate 2. We multiply (1.1) by χ_t and (1.3) by $-\theta^{-1}$ and add. Integration over $\Omega \times (0,t)$, where $t > 0$, yields that

$$\int_0^t \int_\Omega \left(\mu(\theta)\chi_t^2 + \frac{|\nabla\theta|^2}{\theta^2} \right) dx\, d\tau = \int_\Omega \big[\ln\big(\theta(x,t)\big) - \ln\big(\theta_0(x)\big)\big]\, dx$$
$$- \int_0^t \int_\Omega \big(F_1'(\chi) + \beta_2 F_2'(\chi)\big)\chi_t \, dx\, d\tau \leqslant C_4. \tag{3.13}$$

Using (3.4) and (3.12), we conclude from (3.13) that

$$\int_0^t \int_\Omega \big(\chi_t^2 + |\nabla\theta|^2\big)\, dx\, d\tau \leqslant C_5 \quad \forall t \geqslant 0, \tag{3.14}$$

and (3.6) is proved.

Estimate 3. Next, we multiply (1.3) by θ_t and integrate over Ω. Then, for a.e. $t > 0$,

$$\big\|\theta_t(\cdot,t)\big\|_2^2 + \frac{1}{2}\frac{d}{dt}\big\|\nabla\theta(\cdot,t)\big\|_2^2 \leqslant C_6\big(1 + \big\|\theta_t(\cdot,t)\big\|_1\big) \leqslant \frac{1}{2}\big\|\theta_t(\cdot,t)\big\|_2^2 + C_7, \tag{3.15}$$

whence

$$\big\|\theta_t(\cdot,t)\big\|_2^2 + \frac{d}{dt}\big\|\nabla\theta(\cdot,t)\big\|_2^2 \leqslant C_8. \tag{3.16}$$

Thus, combining (3.14) with (3.16), and applying Lemma 3.1 in [10], we can conclude that (3.7) holds.

Estimate 4. Now assume that $\mu(\theta) = \hat{\mu}\theta^{-1}$. Then (1.1) becomes

$$\hat{\mu}\chi_t = -\theta F_1'(\chi) - (\beta_1 + \beta_2\theta)F_2'(\chi) - F_3'(\chi) - w, \tag{3.17}$$

whence, differentiating with respect to t, multiplying by χ_t, and using the fact that $|w_t|$ is bounded, we find that

$$\frac{d}{dt}\chi_t^2 \leqslant C_9\big(1 + |\theta_t|\big) \quad \text{a.e. in } \Omega \times (0,\infty). \tag{3.18}$$

Hence, in view of (3.16),

$$\frac{d}{dt}\big\|\chi_t(\cdot,t)\big\|_2^2 + \frac{d}{dt}\big\|\nabla\theta(\cdot,t)\big\|_2^2 \leqslant C_{10} \quad \text{for a.e. } t > 0. \tag{3.19}$$

Therefore, invoking (3.14), we can infer from Lemma 3.1 in [10] that (3.8) holds. This concludes the proof of the assertion. □

References

[1] P.W. Bates, A. Chmaj, An integrodifferential model for phase transitions: Stationary solutions in higher space dimensions, J. Stat. Phys. 95 (1999) 1119–1139.

[2] P. Colli, J. Sprekels, On a Penrose–Fife model with zero interfacial energy leading to a phase-field system of relaxed Stefan type, Ann. Mat. Pura Appl. 169 (1995) 269–285.

[3] H. Gajewski, On a nonlocal model of non-isothermal phase separation, WIAS Preprint No. 671 (2001). Accepted for publication in Adv. Math. Sci. Appl., in press.

[4] H. Gajewski, K. Zacharias, On a nonlocal phase separation model, WIAS Preprint No. 656 (2001), submitted for publication.
[5] W. Horn, J. Sprekels, S. Zheng, Global existence of smooth solutions to the Penrose–Fife model for Ising ferromagnets, Adv. Math. Sci. Appl. 6 (1996) 227–241.
[6] P. Krejčí, J. Sprekels, Phase-field models with hysteresis, J. Math. Anal. Appl. 252 (2000) 198–219.
[7] P. Krejčí, J. Sprekels, Nonlocal phase-field models for non-isothermal phase transitions and hysteresis, in preparation.
[8] P. Krejčí, J. Sprekels, S. Zheng, Asymptotic behaviour for a phase-field system with hysteresis, J. Differential Equations 175 (2001) 88–107.
[9] O. Penrose, P.C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, Physica D 43 (1990) 44–62.
[10] W. Shen, S. Zheng, On the coupled Cahn–Hilliard equations, Comm. Partial Differential Equations 18 (1993) 701–727.